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A canonical approach to the quantization of the damped harmonic oscillator

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Abstract
We provide a new canonical approach for studying the quantum mechanical damped harmonic oscillator based on the doubling of degrees of freedom approach. Explicit expressions for Lagrangians of the elementary modes of the problem, characterizing both forward and backward time propagation, are given. A Hamiltonian analysis, showing the equivalence with the Lagrangian approach, is also done. Based on this Hamiltonian analysis, the quantization of the model is discussed.

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The damped harmonic oscillator (dho) problem is characterized by the breaking of time-reversal symmetry. A direct Lagrangian formulation is problematic because it leads to explicitly time-dependent Lagrangians [1, 2]. The standard approach [3–5] is to complement the dho by its time-reversed image and work with an effective doubled system. The dynamical group of symmetry of this doubled system is found to be $SU(1, 1)$ but no unitary irreducible representation of the symmetry exists. Time evolution leads out of the Hilbert space of states and a satisfactory quantization can only be achieved in the framework of quantum field theory. This quantization procedure is based on the composite Lagrangian of the effective system resulting from the doubling of degrees of freedom. The lack of individual Lagrangian prescriptions leads to problems in quantization.

We present in this paper a new method of canonical quantization of the dho based on the doubling of degrees of freedom. Explicit expressions of the Lagrangians that characterize dual aspects of the forward and backward time propagations are given. We have shown that the two cases of overdamped and (oscillatory) underdamped motions correspond to distinct regimes characterized by real and complex parameters, respectively, of the constituent Lagrangians. The Hamiltonians corresponding to the complex Lagrangians are found to be pseudo-Hermitian [6]. We have discussed the diagonalization of the complex Hamiltonians pertaining to this regime as a generalization of the Dirac–Heisenberg method of treating the linear harmonic oscillator. The breakdown of time-reversal symmetry is manifested in our analysis by the appearence of pseudo-Hermitian Hamiltonians leading to the time evolution of the individual modes by nonunitary operators. However, exploiting the pseudo-Hermiticity of the individual pieces, we have shown that well behaved states of the composite system are formed.

We begin with a review of the problem of the damped harmonic oscillator (dho). The equation of motion of the one-dimensional damped harmonic oscillator is

$$m\ddot{x} + \gamma\dot{x} + kx = 0. \tag{1}$$

The parameters m, γ and k are independent of time. If the ratio

$$R = \frac{k}{\frac{\gamma^2}{4m}} \tag{2}$$

is greater than 1, the motion is oscillatory with exponentially decaying amplitude. Otherwise, the motion is nonoscillatory, i.e. overdamped. Since system (1) is dissipative, a straightforward Lagrangian description leading to a consistent canonical quantization is not available. To develop a canonical formalism we require to consider (1) along with its time-reversed image [3]

$$m\ddot{y} - \gamma\dot{y} + ky = 0 \tag{3}$$

so that the composite system is conservative. Systems (1) and (3) can be derived from the Lagrangian

$$L = m\dot{x}\dot{y} + \frac{\gamma}{2}(x\dot{y} - \dot{x}y) - kxy \tag{4}$$

where x is the dho coordinate and y corresponds to the time-reversed counterpart. Introducing the hyperbolic coordinates x_1 and x_2 [5] where

$$x = \frac{1}{\sqrt{2}}(x_1 + x_2) \qquad y = \frac{1}{\sqrt{2}}(x_1 - x_2) \tag{5}$$

the above Lagrangian can be written in a compact notation as

$$L = \frac{m}{2}g_{ij}\dot{x}_i\dot{x}_j - \frac{\gamma}{2}\epsilon_{ij}x_i\dot{x}_j - \frac{k}{2}g_{ij}x_ix_j \tag{6}$$

where the pseudo-Eucledian metric g_{ij} is given by $g_{11} = -g_{22} = 1$ and $g_{12} = 0$.

The Lagrangian (6) is invariant under the $SU(1, 1)$ transformation

$$x_i \to x_i + \theta\sigma_{ij}x_j \tag{7}$$

where σ is the first Pauli matrix and θ is an infinitesimal parameter.

The composite Lagrangian (6) is analogous to the general bidimensional oscillator Lagrangian

$$L = \frac{m}{2}\dot{x}_i^2 + \frac{B}{2}\epsilon_{ij}x_i\dot{x}_j - \frac{1}{2}kx_i^2 \tag{8}$$

studied recently [8] in connection with the Landau problem. Here one exploits dual aspects of the rotation symmetry of the problem in analysing it in terms of opposite chiralities [9]. Symmetry of (6) under (7) thus offers a possibility of analysing the composite theory in terms of systems having opposite chiralities w.r.t. the continuous transformations (7).

Accordingly, we introduce the Lagrangian doublet

$$L_{\pm} = \pm\frac{\Gamma}{2}\epsilon_{ij}x_i\dot{x}_j - \frac{k_{\pm}}{2}g_{ij}x_ix_j \tag{9}$$

which are separately invariant under (7). The Noether charges corresponding to the transformations (7) are

$$C_{\pm} = \pm \frac{\Gamma}{2} g_{ij} x_i x_j. \tag{10}$$

Thus systems (9) have opposite 'chiralities' w.r.t. the transformation (7), a fact which motivates their introduction as possible elementary forms of (6).

The synthesis of L_+ and L_- is now done by the soldering formalism which has found applications in various contexts. Duality symmetric electromagnetic actions were constructed [10]; implications in higher dimensional bosonization were discussed [11]; the doublet structure in topologically massive gauge theories was revealed [12]; a host of phenomena in two dimensions were analysed [13]. However, the analysis that is closest in spirit to the one that will be presented here, demonstrated the fusion of two one-dimensional chiral oscillators rotating in opposite directions, into a normal two-dimensional oscillator [9]. Indeed, replacing g_{ij} by δ_{ij} in equation (9) converts it into a doublet of chiral oscillators.

We start from a simple sum

$$L(y, z) = L_+(y) + L_-(z) \tag{11}$$

and consider the gauge transformation

$$\delta y_i = \delta z_i = \Lambda_i(t) \tag{12}$$

where Λ_i are some arbitrary functions of time. Under these transformations the change in L is given by

$$\begin{aligned} \delta L(y, z) &= \delta L_+(y) + \delta L_-(z) \\ &= \Lambda_i (J_i^+(y) + J_i^-(z)) \end{aligned} \tag{13}$$

where the currents are

$$J_i^{\pm}(x) = \pm \Gamma \sigma_{ij} \dot{x}_j - k_{\pm} x_i. \tag{14}$$

The idea is to iteratively modify $L(y, z)$ by suitably introducing auxiliary variables such that the new Lagrangian is invariant under the transformations (12). To this end an auxiliary field B_i transforming as (12),

$$\delta B_i = \Lambda_i \tag{15}$$

is introduced and a modified Lagrangian is constructed as

$$L(y, z, B) = L(y, z) - B_i \big(J_i^+(y) + J_i^-(z)\big) - \tfrac{1}{2}(k_+ + k_-) B_i B_i. \tag{16}$$

This Lagrangian is now invariant under (12) and (15). Since the variable B_i has no independent dynamics, it is eliminated by using its equation of motion. The residual Lagrangian no longer depends on y or z individually but only on the difference $y - z$. Writing this difference as x, the residual Lagrangian reproduces (6) with the identification

$$m = -\frac{\Gamma^2}{(k_+ + k_-)} \qquad \gamma = \frac{\Gamma(k_+ - k_-)}{k_+ + k_-} \qquad k = \frac{k_+ k_-}{(k_+ + k_-)}. \tag{17}$$

The essence of the soldering procedure can be understood also in the following alternative way. Use $x_i = y_i - z_i$ in $L(y, z)$ to eliminate z_i so that

$$L(y, x) = -\frac{k_+}{2} g_{ij} y_i y_j - \frac{\Gamma}{2} \epsilon_{ij} [-2 y_i \dot{x}_j + x_i \dot{x}_j] - \frac{k_-}{2} g_{ij} [y_i y_j - y_i x_j - x_i y_j + x_i x_j]. \tag{18}$$

Since there is no kinetic term for y_i it is really an auxiliary variable. Eliminating y_i from $L(y, x)$ by using its equation of motion we directly arrive at (6) with the correspondence (17).

Note that the opposite chirality of the elementary Lagrangians is crucial in the cancellation of the time derivative of y in (18), which in turn is instrumental in the success of the soldering method.

The identification (17) has an immediate consequence. The ratio (2) is found to be

$$R = \frac{k}{\frac{\gamma^2}{4m}} = 1 - \frac{(k_+ + k_-)^2}{(k_+ - k_-)^2}. \tag{19}$$

For real k_+, k_-, the parameters identified by (17) correspond to an overdamped motion of the dho[3]. Also note that to get the coefficients m and k to be positive we require k_+ and k_- to be of opposite sign, with a suitable choice of their absolute values. Finally, for positive γ, $\Gamma > 0$ is required.

Now the physically more important situation is the underdamped motion of the dho where the motion is oscillatory with decaying amplitude. Here the parameters of (6) are such that the ratio $R > 1$. As already observed, this condition cannot be simulated by the identification (17) for real values of $k_\pm$. However, if k_+ and k_- are continued to complex values so that

$$k_+ = \kappa \qquad k_- = \kappa^* \tag{20}$$

$$R = 1 + \left(\frac{\mathrm{Re}\,\kappa}{\mathrm{Im}\,\kappa}\right)^2 \tag{21}$$

then $R > 1$, which is the required condition for oscillatory motion. Now equation (17) gives

$$m = -\frac{\Gamma^2}{2\mathrm{Re}\,\kappa} \qquad \gamma = \frac{\mathrm{i}\Gamma\mathrm{Im}\,\kappa}{\mathrm{Re}\,\kappa} \qquad k = \frac{|\kappa|^2}{2\mathrm{Re}\,\kappa}. \tag{22}$$

Taking κ of the form

$$\kappa = \kappa_1 + \mathrm{i}\kappa_2 \tag{23}$$

with $\kappa_{1,2}$ positive, we find that Γ must be purely imaginary,

$$\Gamma = -\mathrm{i}g \qquad g > 0 \tag{24}$$

so that the parameters in (22) are positive. Substituting (20) and (24) in (9) we get the elementary modes

$$L_+ = -\mathrm{i}\frac{g}{2}\epsilon_{ij}x_i\dot{x}_j - \frac{\kappa}{2}g_{ij}x_ix_j \tag{25}$$

$$L_- = \mathrm{i}\frac{g}{2}\epsilon_{ij}x_i\dot{x}_j - \frac{\kappa^*}{2}g_{ij}x_ix_j \tag{26}$$

the soldered form of which is the Lagrangian (6) pertaining to the oscillatory limit. Remarkably, the Lagrangians $L_\pm$ are now complex conjugates of each other.

The Lagrangians (25) and (26) both contain information about forward and backward motion in time. To see this, we write L_+ in the form

$$L_+ = -\mathrm{i}gx_1\dot{x}_2 - \frac{\kappa}{2}\left(x_1^2 - x_2^2\right) \tag{27}$$

from which the Euler–Lagrange (EL) equations follow as

$$\mathrm{i}g\dot{x}_2 = -\kappa x_1 \tag{28}$$

$$\mathrm{i}g\dot{x}_1 = -\kappa x_2. \tag{29}$$

[3] See the discussion below (2).

According to equations (22), (23) and (24) we have

$$\frac{\kappa_1}{g} = \Omega \quad \text{and} \quad \frac{\kappa_2}{g} = \frac{\gamma}{2m} \tag{30}$$

where

$$\Omega = \left(\frac{1}{m}\left(k - \frac{\gamma^2}{4m}\right)\right)^{\frac{1}{2}}. \tag{31}$$

The solutions to (28) and (29) are easy to find. Using (30) these solutions can be written in terms of the physical parameters of the dho. Now substituting in (5), we get

$$x = A\,\exp\left(-\frac{\gamma}{2m}t\right)\exp(\mathrm{i}\Omega t) \tag{32}$$

$$y = A\,\exp\left(\frac{\gamma}{2m}t\right)\exp(-\mathrm{i}\Omega t). \tag{33}$$

Clearly, x and y correspond to forward and backward time propagation with reference to the doubling of coordinates (see (1) and (3)). The same solutions also follow from L_-. In this connection it may be observed that Lagrangians structurally similar to (25) and (26) were discussed in [5], as the $m \to 0$ limit of (6). However, it is to be stressed that they are not quite identical because the coefficients of (25) and (26) are completely different from that of the limiting form of (6). This is clearly revealed by the calculation of the friction coefficient presented in [5], which comes out to be different from that of the actual damped oscillator.

It will be instructive to look at the problem from the Hamiltonian approach. The Hamiltonian following from (6) is

$$H = \frac{1}{2m}\left(p_1 - \frac{\gamma}{2}x_2\right)^2 + \frac{k}{2}x_1^2 - \frac{1}{2m}\left(p_2 + \frac{\gamma}{2}x_1\right)^2 - \frac{k}{2}x_2^2 \tag{34}$$

where $p_1 = m\dot{x}_1 + \frac{\gamma}{2}x_2$, $p_2 = -m\dot{x}_2 - \frac{\gamma}{2}x_1$ are the canonical momenta conjugate to x_1 and x_2, respectively. Introduce a canonical transformation from (x_1, x_2, p_1, p_2) to (x_+, x_-, p_+, p_-) where

$$\begin{aligned} p_\pm &= \left(\frac{\omega_\pm}{2m\Omega}\right)^{\frac{1}{2}} p_1 \pm \mathrm{i}\left(\frac{m\Omega\omega_\pm}{2}\right)^{\frac{1}{2}} x_2 \\ x_\pm &= \left(\frac{m\Omega}{2\omega_\pm}\right)^{\frac{1}{2}} x_1 \pm \mathrm{i}\left(\frac{1}{2m\Omega\omega_\pm}\right)^{\frac{1}{2}} p_2. \end{aligned} \tag{35}$$

Such transformations, though involving only real parameters, were used in [12, 14]. Now the composite Hamiltonian diagonalizes as

$$H = H_+ + H_- \tag{36}$$

where

$$H_\pm = \frac{p_\pm^2}{2} + \frac{\omega_\pm^2 x_\pm^2}{2} \tag{37}$$

with the frequencies $\omega_\pm$,

$$\omega_\pm = \Omega \pm \frac{\mathrm{i}\gamma}{2m}. \tag{38}$$

The Hamiltonians $H_\pm$ can be shown to follow from the Lagrangians $L_\pm$. Indeed, the Lagrangian (27) is already in the first-order form. Thus we can read off the Hamiltonian directly,

$$\mathcal{H}_+ = \frac{\kappa}{2}\left(x_1^2 - x_2^2\right) \tag{39}$$

with the symplectic algebra

$$\{x_i, x_j\} = -\frac{\mathrm{i}}{g}\epsilon_{ij}. \tag{40}$$

From (40) we find that $\mathrm{i}gx_1$ is canonically conjugate to x_2. Now by a canonical transformation to the set (x, p_x) defined by

$$x_1 = -\frac{\mathrm{i}}{\sqrt{-\kappa}}p_x \qquad x_2 = \frac{\sqrt{-\kappa}}{g}x \tag{41}$$

the Hamiltonian (39) becomes

$$\mathcal{H}_+ = \left(\frac{p_x^2}{2} + \frac{\omega_+^2 x^2}{2}\right) \tag{42}$$

where we have used equations (23), (30) and (38). The above Hamiltonian coincides with H_+ of (37). Similarly we can derive H_- from L_-. The correspondence between the Lagrangian and Hamiltonian formulations is thus complete.

A question may arise regarding the interpretation of the complex Hamiltonians $H_\pm$ found in the constituent pieces. The first point to note is that they satisfy

$$H_\pm^\dagger = H_\mp. \tag{43}$$

This Hermitian conjugation property corresponds to the time-reversal operation that connects the doubled degrees of freedom of the closed theory. Also, this property manifestly ensures the Hermiticity of the complete Hamiltonian (36).

Although $H_\pm$ are not Hermitian, they are pseudo-Hermitian [6],

$$H_\pm^\dagger = \eta H_\pm \eta^{-1} \tag{44}$$

where η is the PT operator. Such Hamiltonians have occurred in the study of PT-symmetric quantum mechanics [7], in minisuperspace quantum cosmology and other constructions [6]. To prove the condition (44) note that

$$\eta x_i \eta^{-1} = g_{ij}x_j \qquad \eta p_i \eta^{-1} = -g_{ij}p_j. \tag{45}$$

The Hamiltonians $H_\pm$, given by (37), are of the form

$$H = \frac{p^2}{2} + \frac{\omega^2}{2}x^2 \tag{46}$$

where x and p are non-Hermitian and ω is a complex number. Under $\eta = PT$ the operators x and p transform as

$$\eta x \eta^{-1} = x^\dagger \quad \text{and} \quad \eta p \eta^{-1} = -p^\dagger. \tag{47}$$

Now define

$$a = \sqrt{\frac{\omega}{2}}\left(x + \frac{\mathrm{i}p}{\omega}\right) \tag{48}$$

and

$$\tilde{a} = \eta^{-1}a^\dagger\eta = \sqrt{\frac{\omega}{2}}\left(x - \frac{\mathrm{i}p}{\omega}\right). \tag{49}$$

Here $\tilde{a}$ is aptly called the pseudo-Hermitian adjoint of a with respect to η. Writing

$$N = \tilde{a}a \tag{50}$$

we get

$$H = \omega\left(N + \tfrac{1}{2}\right). \tag{51}$$

From the basic commutators between the canonical variables x and p it is easy to derive that

$$\begin{aligned} &[N, a] = -a \\ &[N, \tilde{a}] = \tilde{a}. \end{aligned} \tag{52}$$

Also

$$\eta^{-1} N^\dagger \eta = N. \tag{53}$$

Assume that we can construct a complete bidimensional eigenbasis $\{|\psi_n\rangle, |\phi_n\rangle\}$ diagonalizing N:

$$\begin{aligned} &N|\psi_n\rangle = n|\psi_n\rangle \\ &N^\dagger|\phi_n\rangle = n^*|\phi_n\rangle \\ &\langle\phi_n|\psi_m\rangle = \delta_{nm} \\ &\sum |\phi_n\rangle\langle\psi_n| = \sum |\psi_n\rangle\langle\phi_n| = 1. \end{aligned} \tag{54}$$

Due to (51) this is also the eigenbasis of the Hamiltonian. Now using the commutation relations it can be shown that

$$N(a|\psi_n\rangle) = (n-1)a|\psi_n\rangle. \tag{55}$$

Hence we can write

$$a|\psi_n\rangle = c|\psi_{n-1}\rangle \tag{56}$$

where c is some c-number. Similarly

$$\langle\phi_n|\tilde{a} = d\langle\phi_{n-1}|. \tag{57}$$

The pseudo-Hermiticity of N can be exploited to relate $\eta|\phi_n\rangle$ with $|\psi_n\rangle$ because

$$N\eta|\phi_n\rangle = n\eta|\phi_n\rangle. \tag{58}$$

Using the first equation of (54) we find, up to a phase, the following identification:

$$\eta|\phi_n\rangle = |\psi_n\rangle. \tag{59}$$

The correspondence (59) enables us to reach a crucial result

$$\langle\phi_n|\tilde{a}|\psi_{n-1}\rangle = \langle\phi_{n-1}|a|\psi_n\rangle^* \tag{60}$$

which, along with (56) and (57), gives

$$d = c^*. \tag{61}$$

The last result can be used to show that

$$n = |c|^2. \tag{62}$$

We find that the eigenvalues of N are real and positive. We can also argue that it is integral, otherwise repeated application of a would yield a negative eigenvalue of N. Thus, there exists a state $|0\rangle$ which is annihilated by a:

$$a|0\rangle = 0. \tag{63}$$

Due to (51) this state is the ground state of the Hamiltonian. From the ground state $|0\rangle$ one can develop all the higher energy states by repeated application of $\tilde{a}$.

From the above solution of the eigenvalue problem of (46) we can build the physical states of the composite system by forming direct products. Observe that due to (43) the eigenbasis of H_- will be $\{|\phi_n\rangle, |\psi_n\rangle\}$ if the eigenbasis of H_+ is $\{|\psi_n\rangle, |\phi_n\rangle\}$.

Any formulation of the dho is based on the direct [1] or indirect representation [3–5]. The direct representation leads to Lagrangians having an explicit time dependence; hence these are not very popular. The indirect representation avoids this problem by a doubling of the degrees of freedom. It is called indirect because taking the composite Lagrangian and varying one degree of freedom yields the equation of motion for the other degree (see (4) and its relevant equations of motion). The usual composite Lagrangian, by construction, is two dimensional. It incorporates both forward and backward time propagations. Individual one-dimensional Lagrangians displaying these properties were non-existent.

The new point in our paper is that we have provided explicit one-dimensional Lagrangians (equations (25) and (26)) that characterize both forward and backward time propagations. Although structurally these Lagrangians look two dimensional, the symplectic structure effectively reduces to one dimension. Moreover we showed that a combination of these Lagrangians led to (6). In this sense these Lagrangians are more fundamental. Also, they cannot be obtained by taking the simple $m \to 0$ limit of (6). In the region of the parameter space which corresponds to the damped oscillatory motion, the parameters of the constituent Lagrangians were complex valued. Also, these Lagrangians were complex conjugates of one another. Because of this property, the resulting Hamiltonians were complex valued, satisfying the requirements of pseudo-Hermiticity. This pseudo-Hermiticity was exploited to diagonalize the individual Hamiltonians. Based on this, an alternative quantization of the damped harmonic oscillator was indicated.

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